In the hypersonic limit, terms of $0[\gamma - 1/\gamma]$ $[h\delta/H\delta]$ may be neglected with respect to terms of order 1 The energy equation can now be written

$$\frac{1}{Pr}\frac{\partial}{\partial\eta}\left\{N_0 g_1' + \frac{T_w}{g_w}\frac{dN}{dT}(g_1 - 2f_0'f_1')g_0'\right\} + (f_0 g_1)' + \frac{2\partial}{\partial\eta}\left\{N_0\left(1 - \frac{1}{Pr}\right)(f_1'f_0'' + f_0'f_1'')\right\} + 2\frac{\partial}{\partial\eta}\left\{\frac{T_w}{g_w}\frac{dN}{dT}(g_1 - 2f_0'f_1')(f_0'f_0'')\right\} = 0 \quad (4)$$

Equation (4) can be integrated and the constant of integration set equal to zero in order to insure the exponential decay of the profiles, as well as the boundedness of the various boundary layer thicknesses The resulting equation can be evaluated at $\eta = 0$:

$$\frac{1}{Pr}\left\{g_1'(0) + \left(\frac{T_w}{g_w}\frac{dN}{dT}\right)_{T=T_w}[g_1(0)g_0'(0)]\right\} + 2\left(1 - \frac{1}{Pr}\right)[f_1'(0)f_0''(0)'] = 0 \quad (5)$$

The energy transfer to a wall in a slip flow is $[k(\partial T/\partial y)_0] + (\tau u)_0 \equiv q$ In terms of our η, ξ variables, this corresponds to

$$q = \frac{H_\delta \rho(0)\mu(0)\mu_\delta}{(2\xi)^{1/2}Pr}[g(0) - 2f'(0)f''(0)] + \frac{H_\delta\rho(0)\mu(0)2f'(0)f''(0)}{(2\xi)^{1/2}} \quad (6)$$

In terms of our expansion parameter ϵ,

$$q = \frac{H_\delta p_0(1 + \epsilon p_1)(1 - \epsilon p_1)}{2\xi_0^{1/2}Pr}\frac{\mu_w}{RT_w}\left[1 + \left(\frac{T_w}{g_w}\frac{dN}{dT}\right)_{T=T_w}(g_1)\epsilon\right] \times [g_0'(0) + \epsilon g_1'(0) - 2\epsilon f_1'(0)f_0''(0) + 2Pr\epsilon f_1'(0)f_0''(0)] \quad (7a)$$

or

$$q = q_0\left[1 + \epsilon\left(\frac{T_w}{g_w}\frac{dN}{dT}\right)_{T=T_w} g_1(0) + \epsilon\frac{g_1'(0)}{g_0'(0)} + 2(Pr - 1)\epsilon \times \frac{f_1'(0)f_0''(0)}{g_0'(0)}\right] \quad (7b)$$

From Eq (5), the coefficient of ϵ is equal to 0 Thus $q_{\text{lip}} = q_{\text{no slip}}$

As noted earlier, this result has been obtained before by Maslen[2] and has been the center of some controversy among Rott,[5] Shen,[6] and Maslen[7] It would appear that this especially simple result, independent of the magnitudes of Pr, a_v, or a_T, hinges on the correct application of the transformation $(\partial\eta/\partial_y)_{y=0} = \rho(0)u_\delta/(2\xi)^{1/2}$ rather than $\rho(w)u_\delta/2^{1/2}\xi$ For $(\gamma - 1)/\gamma \to 0$, the solutions of Eqs (2a) and (2b) are

$$f_1' = f_0''(\eta)\left\{a_V - [a_T - a_V]g_0'(0)\int_0^\eta\left[\frac{N(\eta)f_0''(\eta)}{N(0)f_0''(0)}\right]^{Pr} \times \left[\frac{dN_0}{N_0 dT_0}\frac{T_w}{g_w}\right]d\eta\right\} \quad (8a)$$

$$g_1 = g_0'(\eta)\left\{a_V - [a_T - a_V]g_0'(0)\int_0^\eta\left[\frac{N_0(\eta)f''(\eta)}{N_0(0)f'(0)}\right]^{Pr} \times \left[\frac{dN_0}{N_0 dT_0}\right]\frac{T_w}{g_w}d\eta\right\} + [a_T - a_V]g_0'(0)\left[\frac{N_0(\eta)f_0''(\eta)}{N(0)f_0''(0)}\right]^{Pr} \quad (8b)$$

An approximate estimate of the effect of slip on induced pressure can be obtained quite simply for this limiting hypersonic case, $N = 1$ and $Pr = 1$ The displacement thicknes- can be written

$$\delta^* = \delta_0^*[1 + \epsilon\delta_1/\delta_0] \quad (9a)$$

where

$$\delta_1 = \int_0^\infty (g_1 - 2f_0'f_1')d\eta \quad (9b)$$

$$\delta_0 = \int_0^\infty (g_0 - f_0'^2)d\eta \quad (9c)$$

Thus

$$\delta_1 \approx -a_v g_w + [a_T - a\ ][1 - g_w] \quad (10)$$

From the tangent wedge formula, $p_1 = \frac{4}{3}(\delta_1/\delta_0)$, or using the Crocco integral and the Blasius' results,

$$p_1 \approx \frac{2^{1/2}(-a_v g_w + [a_T - a_v][1 - g_w])}{1\ 73 g_w + 0\ 664}\frac{4}{3} \quad (11)$$

This result implies that slip effects on a cold wall will tend to increase the induced pressure from the no-slip values, whereas slip effects on a near adiabatic wall, where $g_w \sim 1$, will lower the induced pressure The latter effect has been observed, while the former disagrees with currently available data Our analysis, although incomplete and approximate, does suggest that slip alone is not sufficient to cause strong interaction theory to agree with experiment as the leading edge is approached, at least for a cold wall

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Reply by Author to J Aroesty

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The writer does not think that there are any differences of opinion between Aroesty and himself, except perhaps in what we imply in our use of words like "slip flow" and "rarefaction effects," and certainly he agrees with him that the leading edge region cannot be treated adequately merely by applying slip boundary conditions to the hypersonic boundary layer equations The main purpose of the note was to suggest that the region of the leading edge flow, where the predictions of hypersonic boundary layer theory begin to depart from the experiment, might be characterized by a particular Knudsen number based on properties of the hypersonic boundary layer The fluid-dynamic phenomena responsible for these departures most likely include the merging of a thick shock wave with a completely viscous layer, as well as slip at the wall, and the flow probably resembles more the model considered by Oguchi[1] than a slipping hypersonic boundary

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layer In using the terms "slip flow" and "rarefaction effects," the writer had in mind the combination of all of these phenomena, and for this reason it was not unreasonable to suppose that these effects might correlate better with a mean free path behind the shock wave rather than at the wall Whether or not these "rarefaction phenomena," in the sense that the writer used the term, are susceptible to analysis within the framework of Navier-Stokes theory is still an open question It is his opinion, evidently shared by Aroesty, that the resources of the Navier-Stokes equations are not yet exhausted

Reference

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Flow in Contracting Ducts

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THERE are a few points of interest arising from the paper by Szczeniowski [1] He draws attention to two solutions for the two-dimensional irrotational, incompressible flow through contracting ducts These are, of course, two of many such existing solutions, some of which have been critically compared by Moretti [2]

It is of interest to examine further the second solution quoted by Szczeniowski [1] The expression for the ordinates in the $z \equiv x + iy$ plane in terms of the complex potential $w \equiv \varphi + i\psi$ is given as

$$e^{\;/a} = e^{\mu(w/aU)} + e^{w/aU}$$

Taking, for algebraic simplicity, both a and U as unity, differentiating and rearranging results in

$$w = \frac{1}{\mu - 1} \log\left(\frac{dw}{dz} - 1\right) - \frac{1}{\mu - 1} \log\left(\frac{dw}{dz} - \frac{1}{\mu}\right) + \frac{i\pi - \log\mu}{\mu - 1}$$

where $i\pi$ is taken as the principle value of $\log(-1)$ Now the dw/dz plane is the hodograph plane, for $dw/dz = u - iv$ Thus, in this plane the flow is from a source of strength $1/(\mu - 1)$ at the point (1,0) to a sink of the same strength at the point $(1/\mu,0)$ This solution has also been described by Libby and Reiss[3] and the present writer [4] The outermost streamline along which the velocity rises monotonically is the circle whose diameter lies along the x axis Along the axis corresponding to the duct centerline, $1/\mu \leq dw/dz \leq 1$ (note that the flow is backward in the duct), and so $+\infty \geq \varphi \geq -\infty$ and $\psi = 2\pi/(\mu - 1)$ The outermost streamline has a stream function value that is less than this by $(\pi/2)/(\mu - 1)$ and thus, along it, $\psi = (3\pi/2)/(\mu - 1)$ for all values of μ $(\mu > 1)$

It does not appear possible to obtain any exceptions to the result that the velocity distribution in contractions becomes uniform only at infinity up- and downstream This can be seen from inspection of the hodograph plane as shown in Fig 1 (The sketch is of the logarithmic hodograph plane, $\log(dw/dz) = \log q - i\theta$)

In this plane a uniform flow transforms to a single point, and so the upstream and downstream flows, when uniform, transform, for any boundary shape, to the two points marked

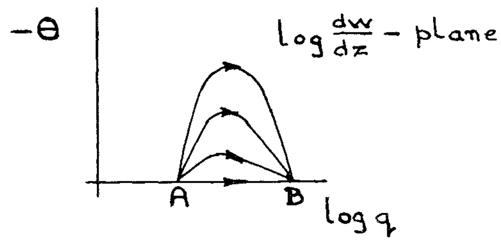

Fig 1

A and B The two uniform flows thus correspond to the origins of a source and sink flow at which φ has the values $-\infty$ and $+\infty$, respectively Transformation back into the z plane by integration[5] gives the corresponding x values as $\mp\infty$

It seems difficult to imagine a flow pattern in the logarithmic hodograph plane where the outer streamline gives a continuous increase in $\log q$, whereas an intermediate one has a maximum and a minimum value The flow along the axis always rises monotonically whatever the boundary shape However, in the real flow a monotonically rising velocity is required only along the boundary

Real contracting ducts are finite in length, usually adjoining parallel wall ducts upstream and downstream As Goldstein showed,[6] two regions of falling velocity gradient occur along the duct boundary There are two general cases:

1) The first case is one in which adverse gradients can be restricted to occur along the parallel walls The boundary in the logarithmic hodograph plane is as sketched in Fig 2, lettered $A\ C\ D\ B$ and marked I, A and B being the origins of the source and the sink flow, respectively

2) The second case is one in which adverse gradients can be restricted to occur along the curved wall Such a boundary is also sketched in Fig 2, lettered $A\ C'\ C\ D\ D'\ B$ and marked II, I, III; cuts occur along AC' and BD' † The resulting curved boundary has three points of inflexion in the z plane, as sketched in Fig 3

One can, of course, specify that condition 1 applies at one end of the duct and condition 2 applies at the other Although the velocity still becomes uniform only in a finite duct at infinity up- and downstream, it approaches its asymptotic value more rapidly than in the infinite length contraction This can be illustrated by considering an approximation to the flow sketched in Fig 2, which is provided by the flow sketched in Fig 4 Here AD and CE are straight lines going to infinity

The flow then consists of a source at A together with its image source; thus, writing the ratio of the velocities at A and B as n, the flow is given by

$$w = \log \log \frac{dw}{dz} + \log\left[\log \frac{dw}{dz} - 2 \log n\right]$$

When $\theta = 0$, $\sin\psi = 0$ and then, for $0 \leq \log q \leq +\infty$,

$$0 < e^{\varphi} \cos\psi \leq +\infty$$

and for $-\log n \leq \log q \leq 0$,

$$-(\log n)^2 \leq e^{\varphi} \cos\psi \leq 0$$

Thus, putting $\psi = 0$ along $A\ D$ gives $\psi = \pi$ along $A\ C$ and also along $A\ D$

$$e^{\varphi} = \log q \log n^2 q$$

and along $A\ C$

$$e^{\varphi} = -\log q \log n^2 q$$

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† These cuts can extend from A partly toward C and from B partly toward D